

The QCD observables expansion over the scheme-independent two-loop coupling constant powers, the scheme dependence reduction

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Abstract

The method suggested in this paper allows to express the n -th order renorm-group equation solutions over the powers of the two-loop solution, that can be obtained explicitly in terms of the Lambert function [1, 2, 3]. On the one hand this expansion helps to get more reliable theoretical predictions, on the other hand the scheme dependence problem can be understood better. When using this method, Stevenson scheme invariant expressions [4] can be obtained easily, the scheme dependence emerging from the perturbative series truncation can be estimated and reduced. The ‘optimal’ choice of the scale parameter allows to have at the three-loop level the scheme dependence magnitude corresponding to the four-loop level etc. The new criterion, principally different from the Raćzka criterion [5], is introduced.

1 Introduction

The renorm-group equation (RGE) for the QCD coupling constant $\bar{\alpha}_s$

$$-x \frac{\partial a}{\partial x} = a^2 + b_1 a^3 + b_2 a^4 + \dots, \quad a = \frac{\beta_0}{4\pi} \bar{\alpha}_s, \quad b_n = \frac{\beta_n}{\beta_0^n}. \quad (1)$$

can be solved analytically at the one-loop level (leading order) $a^{(1)}(x) = \frac{1}{\ln x}$, the 2nd-order (next-to-leading order) RGE solution can be expressed in terms of the Lambert function [1, 2]

$$\tilde{a}(x) = -\frac{1}{b_1} \frac{1}{1 + W(z)} \quad , \quad z \equiv -\frac{1}{e} x^{-\frac{1}{b_1}}. \quad (2)$$

The W Lambert function is defined through a transcendental equation $W(z)e^{W(z)} = z$. Here and further $x = \frac{Q^2}{\Lambda^2}$, Λ is the scale parameter defining unphysical singularities positions. At the two-loop level unphysical cut emerges when $Q^2 < \Lambda^2$.

For the 3rd order RGE the Padé-approximated β -function [6] can be used and the solution can be also obtained in terms of the Lambert function. However, without Padé approximation even at the three-loop case we can’t get explicitly the RGE solution for both coupling constant and observables.

Another problem is observables unphysical scheme dependence that is related with the multi-loop terms uncertainty. It originates the theoretical uncertainty of physical results. The using of different renorm-schemes (RS) leads to various results and there are not any reasons to consider one of these schemes as the most preferable. The natural way to reduce this uncertainty is to reduce the scheme dependence.

As an observable example the process of e^+e^- -annihilation into hadrons is used in this paper. This process is rather interesting when analysing the scheme dependence [7].

The method suggested in this paper is based on the expansion of the n-th order RGE solutions over the two-loop coupling constant (2) powers. This approach proves to be very useful to analyse the problem of observables scheme dependence, to get perturbative coefficient invariant combinations, to estimate the theoretical uncertainty (the new criteria was elaborated for this purpose). But the most important consequence is the possibility of reducing this uncertainty.

2 The expansion of multi-loop RGE solutions over the two-loop solution powers

We used the expansion of the n-th order RGE solution over the powers of some known solution (obtained in the certain RS). The attempt to obtain such an expansion over the one-loop function powers can not be successful, because even the 2nd order RGE solution possesses log log singularities that couldn't be expanded over the powers of $a^{(1)}(x) = \frac{1}{\ln x}$. It can be easily understood also with the Lambert function consideration. The two-loop coupling constant expressed as a function of the one-loop constant has an evident singularity in the vicinity of $a^{(1)} = 0$, originating from the Lambert function argument $z = -e^{\frac{1}{b_1 a^{(1)}}}$. Let us begin with the assumption of the expansion over the two-loop function powers existing

$$a^{(n)} = \sum_i k_i \tilde{a}^i. \quad - \frac{\partial \tilde{a}}{\partial t} = \tilde{a}^2 + b_1 \tilde{a}^3, \quad t = \ln x, \quad (3)$$

i.e. $a^{(n)}$ is n-th order RGE solution. To obtain k_i values let us consider

$$\sum_i k_i \partial_t \tilde{a}^i = \sum_i k_i (\tilde{a}^2 + b_1 \tilde{a}^3) = \left(\sum_i k_i \tilde{a}^i \right)^2 + b_1 \left(\sum_i k_i \tilde{a}^i \right)^3 + \dots \quad (4)$$

From the $\tilde{a}(x)$ powers coefficients equality follows

$$k_1 = 1; k_2 = ?; k_3 = k_2^2 + b_1 k_2 + b_2; k_4 = k_2^3 + \frac{5}{2} b_1 k_2^2 + 3b_2 k_2 + \frac{1}{2} b_3.$$

k_2 coefficient is arbitrary, it corresponds to the arbitrariness of the scale parameter choice. Indeed, if we express some RGE solution through the certain two-loop solution we should have the parameter that permits to obtain solutions with a different scale parameter.

Researching of the k_i asymptotic properties is complicated because of the involved recurrent formula

$$k_i = k_{i-1} b_1 \left(-1 + \frac{3}{i-2} \right) + k_{i-2} b_2 \frac{3b_2}{i-2} + \dots (\text{for } i > 4).$$

Suppose $k_2 = 0$ (that corresponds to the natural choice, when the multi-loop coupling constant and the two-loop constant have the same scale parameter). So, k_i values are:

$$\begin{aligned} k_3 &= b_2; k_4 = \frac{1}{2} b_3; k_5 = \frac{5}{3} b_2^2 - \frac{1}{6} b_1 b_3 + \frac{1}{3} b_4 \\ k_6 &= -\frac{1}{12} b_1 b_2^2 + \frac{1}{12} b_1^2 b_3 + 2b_2 b_3 - \frac{1}{6} b_1 b_4 + \frac{1}{4} b_5 \\ k_7 &= \frac{16}{5} b_2^3 - \frac{4}{5} b_1 b_2 b_3 + \frac{1}{20} b_1^2 b_2^2 - \frac{1}{20} b_1^3 b_3 + \frac{11}{20} b_2^2 b_3 + \frac{1}{10} b_1^2 b_4 - \frac{3}{20} b_1 b_5 + \frac{1}{5} b_6. \end{aligned}$$

3 The observables scheme dependence

3.1 The RS invariant expressions

Consider the case when some observable f is expressed over the powers of coupling constant perturbatively

$$f(x) = a(x) \left(1 + \sum_{i=1} d_i a(x)^i \right), \quad (5)$$

The renormalization scheme changing affects the expansion coefficients, so does RGE and the coupling constant. The natural physical demand is the observable scheme independence. Initially, there was the expansion over the scheme dependent function powers with the scheme dependent coefficients, but expressing $a(x)$ as the two-loop function power series (3) we obtain the expansion over the scheme-independent two-loop coupling constant

$$\begin{aligned} f(x) = & \tilde{a}(x) + (k_2 + d_1 k_1^2) \tilde{a}^2(x) + (k_3 + 2d_1 k_1 k_2 + d_2 k_1^3) \tilde{a}^3(x) + \\ & + (k_4 + d_1 (2k_1 k_3 + k_2^2) + 3d_2 k_1^2 k_2 + d_3 k_1^4) \tilde{a}^4(x) + \dots \end{aligned} \quad (6)$$

This leads to an invariance of this expansion coefficients. k_2 corresponds to the scale parameter Λ changing $k_2 = -\ln \frac{\Lambda}{\Lambda_0}$. Using the expressions for k_i we can get invariant combinations

$$\rho_1 = d_1 - \ln \frac{\Lambda}{\Lambda_0} \quad (7)$$

$$\rho_2 = \beta_2 + d_2 + \left(\ln \frac{\Lambda}{\Lambda_0} \right)^2 + (-\beta_1 + 2d_1) \ln \frac{\Lambda}{\Lambda_0} \quad (8)$$

$$\begin{aligned} \rho_3 = & \frac{1}{2} \beta_3 + d_3 + 2\beta_2 d_1 + (-3d_2 - 3\beta_2 - 2\beta_2 d_1) \ln \frac{\Lambda}{\Lambda_0} + \left(3d_1 + \frac{5}{2} \beta_1 \right) \left(\ln \frac{\Lambda}{\Lambda_0} \right)^2 + \left(\ln \frac{\Lambda}{\Lambda_0} \right)^3 \\ & \dots \end{aligned} \quad (9)$$

These combinations are invariant after the defining of the parameter Λ_0 certain value. The most prudent way is to make arbitrary parameter Λ_0 close to the scale parameter physical values. Further we suppose $\Lambda_0 = \Lambda_{\overline{MS}}$. Substituting $d_1 - \rho_1$ for $\ln \frac{\Lambda}{\Lambda_0}$ we get

$$\rho_2 = -d_1^2 - \beta_1 d_1 + \beta_2 + d_2 + \beta_1 \rho_1 + \rho_1^2. \quad (10)$$

Similarly,

$$\begin{aligned} & -d_1^2 - \beta_1 d_1 - \frac{1}{4} \beta_1^2 + \beta_2 + d_2 - inv. \\ & -d_1^2 - \beta_1 d_1 + \beta_2 + d_2 - inv. \end{aligned}$$

The first invariant combination was obtained by Stevenson [4, 8], the second one was used by Račka for introducing of the scheme dependence criteria. These three expression differs by scheme invariant terms. However, the coefficients ρ_1, ρ_2, \dots has the definite status. They are not just invariants but the coefficients of observable expansion over the powers of the scheme invariant two-loop coupling constant

$$f(x) = \tilde{a}(x) + \rho_1 \tilde{a}^2(x) + \rho_2 \tilde{a}^3(x) + \dots \quad (11)$$

3.2 The scheme dependence estimate

The changing of RS affects the scale parameter value according to the following precise expression [4, 9]. When the new coupling constant is defined as the power series of the old one

$$a' = a + q_1 a^2 + \dots, \text{ then}$$

$$\Lambda' = e^{\frac{q_1}{\beta_0}} \Lambda, \beta_0 = \frac{9}{4}. \quad (12)$$

That is true only for the precise expressions when all perturbative terms are summed. But the formula (12) is inapplicable for the truncated series, its using leads to the scheme dependence augmenting.

Consider the three-loop case when $\beta_1, \beta_2, d_1, d_2$ are known, we can also consider Λ_0 as defined. All multi-loop coefficients are unknown, so we suppose them to be zero. In order to provide ρ_1 and ρ_2 invariance d_1 and d_2 can be expressed as

$$d_1 = \rho_1 + \ln \frac{\Lambda}{\Lambda_0}$$

$$d_2 = \rho_2 - \left(\ln \frac{\Lambda}{\Lambda_0} \right)^2 + \beta_1 \ln \frac{\Lambda}{\Lambda_0} - \beta_2 + 2d_1 \ln \frac{\Lambda}{\Lambda_0}.$$

ρ_3 can not be invariant since we suppose $\beta_3, d_3 = 0$. That is here, where we lose the observable invariance that results from the perturbative series truncation. ρ_3 scheme dependence can be considered as the measure of the total observable scheme dependence

$$\rho_3(\Lambda, \beta_2) = 2\rho_1\beta_2 + (-2\beta_2 - 3\rho_2 + 2\rho_1\beta_1) \ln \frac{\Lambda}{\Lambda_0} + \left(-3\rho_1 - \frac{5}{2}\beta_1 \right) \left(\ln \frac{\Lambda}{\Lambda_0} \right)^2 + \left(\ln \frac{\Lambda}{\Lambda_0} \right)^3.$$

The observable modification when changing the RS can be expressed

$$\Delta f = \Delta \rho_3 \left(\ln \frac{\Lambda}{\Lambda_0}, \beta_2 \right) \tilde{a}^4 + \Delta \rho_4 \left(\ln \frac{\Lambda}{\Lambda_0}, \beta_2 \right) \tilde{a}^5 + \dots$$

$$\Delta \rho_3 = \Delta \beta_2 \frac{d\rho_3}{d\beta_2} + \Delta \left(\ln \frac{\Lambda}{\Lambda_0} \right) \frac{d\rho_3}{d \ln \frac{\Lambda}{\Lambda_0}}.$$

Two values

$$'_1 = \frac{d\rho_3}{d\beta_2} = 2\rho_1 - 2 \ln \frac{\Lambda}{\Lambda_0},$$

$$C_2 = \frac{d\rho_3}{d \ln \frac{\Lambda}{\Lambda_0}} = -2\beta_2 - 3\rho_2 + 2\rho_1\beta_1 + \ln \frac{\Lambda}{\Lambda_0} (-6\rho_1 - 5\beta_1) + 3 \ln \frac{\Lambda}{\Lambda_0}^2$$

characterise the scheme dependence. When both values are equal zero the observable modification is proportional to the fifth power of the coupling constant.

3.3 The ‘optimal’ choice of the scale parameter value

Let us consider again the three-loop case, when $\beta_1, \beta_2, d_1, d_2$ and Λ_0 are defined. The $\rho_1(\Lambda, d_1)$ and $\rho_2(\Lambda, d_1, d_2, \beta_2)$ invariance allows to present d_1 and d_2 as $d_1(\Lambda)$, $d_2(\Lambda, \beta_2)$. Four parameters $\beta_2, d_1, d_2, \ln \frac{\Lambda}{\Lambda_0}$ that characterise the scheme at the three-loop level are related by two equations, so we could exclude instead, for example, d_1 and β_2 .

The scheme dependence of ρ_3 can be got over by choosing the scale parameter. Let us require the invariance of $\rho_3(\Lambda, d_1, d_2, \beta_2, d_3 = 0, \beta_3 = 0)$, this leads to the defining of the scale parameter as a function of β_2 . Thus, the scheme dependence comes from $\rho_4(\beta_2)$ and it can not be eliminated at the three-loop level.

The method presented in this paper provides the ρ_3 invariance. The observable variation, when the current RS is modified, can be written as

$$\Delta f = (C\Delta\beta_2 + O(\Delta\beta_2))\tilde{a}^5 + O(a^6),$$

$$C = \frac{d\rho_4}{d\beta_2} - \frac{d\rho_4}{d\ln\frac{\Lambda}{\Lambda_0}} \frac{\frac{d\rho_3}{d\beta_2}}{\frac{d\rho_3}{d\ln\frac{\Lambda}{\Lambda_0}}} = -\frac{8}{3}\beta_2 + \rho_2 + 2\rho_1 \frac{-6\rho_1 - 3\beta_1\beta_2 + 3\rho_2\beta_1 + 8\rho_1\beta_2}{-2\beta_2 + 3\rho_2 + 2\rho_1\beta_1}. \quad (13)$$

So, at the three-loop level theoretical uncertainty can be reduced to the a^5 order. This means the scheme dependence reduction comparing to the case, when the prescription (12) is used.

The new criterion of the preferable RS is the minimal C value. Its examination for some observables is presented in the next section.

3.4 The RS new classification

The method suggested here let us reduce the scheme dependence. However, it can not eliminate it, so it is interesting to make some hierarchy of different RS using the new criterion. For this purpose we should know ρ_1 , ρ_2 and β_2 values.

As the first observable example we chose the Adler D -function

	d_1	d_1^{opt}	d_2	d_2^{opt}	b_2	$\frac{\Lambda}{\Lambda_{\overline{MS}}}$	$\frac{\Lambda^{opt}}{\Lambda_{\overline{MS}}}$	C_d
\overline{MS}	0.73	0.73	1.26	1.26	0.88	1	1	4.07
PMS	0	0.92	-0.52	-0.52	1.55	0.70	1.1	2.28
ECH	0	0.77	0	1.20	1.03	0.70	1.02	3.70
$'tH$	-1.47	0.54	1.93	1.75	0	0.37	0.91	6.07
V	-0.048	2.08	-4.17	1.81	5.17	0.68	1.96	-15.2

The Adler function scheme dependence was studied earlier [10, 11]. In this paper the optimal Λ value was obtained for the \overline{MS} -scheme, for the PMS¹-optimized [4] scheme, the effective charge approach (ECH) of Grunberg [12, 13], the t' Hooft RS and V-scheme.

The examination of the e^+e^- -annihilation into hadrons ratio $R(s)$ is presented in the next table

	r_1	r_1^{opt}	r_2	r_2^{opt}	b_2	$\frac{\Lambda}{\Lambda_{\overline{MS}}}$	$\frac{\Lambda^{opt}}{\Lambda_{\overline{MS}}}$	C_r
\overline{MS}	0.73	0.73	1.26	1.26	0.88	1	1	-5.81
PMS	0.73	0.57	-3.81	-3.04	1.55	0.70	0.92	-7.87
ECH	0	0.68	0	-3.28	-2.27	0.70	0.976	-6.26
$'tH$	-1.47	-3.90	-1.35	10.45	0	0.37	0.094	-118
V	-0.048	0.29	-7.47	-7.12	5.17	0.68	0.80	-18.6

The method can be easily generalised onto other observables such as the τ -decay ratio, the Bjorken sum rule etc.

¹PMS stands for the principle of minimal sensitivity

The new criterion values obtained above can not serve for a definite choice of the preferable RS. Thus, the PMS is the optimal RS for the Adler-function among the RS set considered, but the $R(x)$ -function lowest criterion absolute value is for the \overline{MS} -scheme. However, \overline{MS} , PMS and ECH have approximately the same criterion values for both cases and other two schemes have very large values. We can conclude that the new criterion C can not be considered as a universal value characterising the RS but it is suitable for ‘bad’ schemes sorting out.

4 Conclusions

The method suggested in this paper allows to express the renorm-group equation solutions over the powers of the two-loop one. This can be very useful for some numerical estimates, especially after the analyticization procedure applying [14, 15, 16].

This method helps to reduce the scheme dependence, to analyse the problem better and get some estimates of this dependence value. The only value that is adjusted is the scale parameter Λ . We can conclude that the scheme dependence can be minimised just by the ‘optimal’ revaluation of the experimentally extracted value Λ without any additional requirements.

The idea about the observable expansion over the two-loop coupling constant is interesting to compare with the observable expansion in the recently published by C. Maxwell [17] paper. However, we used another way to obtain this expansion, results obtained in this paper are different too.

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